ELSEVIER
Journal of Geometry and Physics 44 (2003) 475–480
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

A negative answer to a Rieffel's question on the behavior of K-groups under strict deformation quantization

Jae-Kwan Shim

Department of Industrial Applied Math, Kyungpook National University, Taegu, South Korea

Received 24 October 2001; received in revised form 14 January 2002

Abstract

In this paper, we address one of the questions raised by Rieffel in his collection of questions on deformation quantization. The question is whether the K-theory groups remain the same under flabby strict deformation quantizations. By "deforming" the question slightly, we produce a negative answer to the question.

MSC: Primary 53D55; 81S

Keywords: Flabby deformation quantization; K-theory; Non-commutative tori

Subj. Class:Quantum mechanics

In his collection of questions on deformation quantization [10], Rieffel asked the following: "Are the K-groups of the C^*-algebra completions of the algebras of any flabby strict deformation quantization all isomorphic?" Up to my knowledge, the question is still open. But this paper will show that the answer is negative if we ask the same question for the case of orbifolds.

Definition 1 (Rieffel [10]). Let $(M, \{\cdot;\cdot\})$ be a Poisson manifold. A strict deformation quantization of M in the direction of $\{\cdot,\cdot\}$ is a dense $*$-algebra A of $C^\infty(M)$ which is closed under the Poisson bracket, together with a closed subset I of the real line containing 0 as a non-isolated point, and for each $\hbar \in I$ an associative product $*\hbar$, an involution $*\hbar$, and a pre-C^*-norm $\|\cdot\|_\hbar$ on A, which for $\hbar = 0$ are the original pointwise multiplication, complex

E-mail address: jshim@semi.korea.ac.kr (J.-K. Shim).

0393-0440/02/$ – see front matter
PII: S0393-0440(02)00033-5

conjugation, and supremum norm, respectively, and such that

(1) $\{\overline{A_\hbar}\}_{\hbar\in I}$ forms a continuous fields of C^*-algebras over I, where $\overline{A_\hbar}$ is the C^*-completion of $A_\hbar = (A, \|\cdot\|_\hbar)$,
(2) for $f, g \in A$

$$\left\| \frac{f *_\hbar g - g *_\hbar f}{\sqrt{-1}\hbar} - \{f, g\} \right\|_\hbar \to 0 \text{ as } \hbar \to 0.$$

This definition still makes sense for a simple Poisson orbifold M/Γ, where Γ is a finite group acting on M and the action preserves the Poisson bracket. A function on M/Γ is just a function on M which is constant on each orbit. A function f on M/Γ is defined to be smooth if it is smooth as a function on M.

Definition 2. A strict deformation quantization is *flabby* if A as above, contains $C_c^\infty(M)$, the algebra of smooth functions of compact support on M. This notion also makes sense for M/Γ as above.

There is more algebraic version of deformation quantization, called the *formal deformation quantization*. A formal deformation quantization of M is defined as an associative algebra structure $*$ on $C^*(M)[[\hbar]]$ ($\hbar$ is a formal letter) such that, for $f, g \in C^*(M)$

$$f * g = fg + \frac{\sqrt{-1}}{2}\{f, g\}\hbar + B_2(f, g)\hbar^2 + B_3(f, g)\hbar^3 + \cdots,$$

where B_i's are bidifferential operators. Using ideas from string theory, Kontsevich [6] proved that any Poisson manifold is formally deformally deformation quantizable.

Now, we consider the example of a strict deformation quantization of tori [8]. We use real coordinates $(x_1, \dots, x_n)$ for the n-torus T^n, viewing T^n as $\mathbb{R}^n/\mathbb{Z}^n$. Any real skew-symmetric matrix Θ defines a Poisson bracket on $C^\infty(T^n)$

$$\{f, g\} := \sum_{j,k} \theta_{jk} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_k} \quad \text{for } f, g \in C^\infty(T^n).$$

The Fourier transform $\mathcal{F}$ maps $C^\infty(T^n)$ to $S(\mathbb{Z}^n)$, the space of complex-valued Schwartz functions. Recall that $\mathcal{F}(f) \in S(\mathbb{Z}^n)$, $f \in C^\infty(T^n)$, is defined as follows, for $p \in \mathbb{Z}^n$

$$\hat{f}(p) := \mathcal{F}(f)(p) = \int_{T^n} \exp(2\pi\sqrt{-1}x \cdot p) f(x)\, \mathrm{d}x,$$

where $\mathrm{d}x$ is the Haar measure with $\int_{T^n} 1\, \mathrm{d}x$. $\mathcal{F}$ is invertible, and its inverse is given by

$$\phi \to \sum_{p\in\mathbb{Z}^n} \phi(p) \exp(2\pi\sqrt{-1}p \cdot x).$$

$\mathcal{F}$ carries the Poisson bracket to

$$\{\phi, \psi\}(p) = -4\pi^2 \sum_q \phi(q)\psi(p-q)\gamma(q, p-q)$$

for $\phi, \psi \in S(\mathbb{Z})^n$, where

$$\gamma(p,q) = \sum_{j,k} \theta_{jk} p_j q_k.$$

For any $\hbar \in \mathbb{R}$, we define a function $\sigma_\hbar$ on $\mathbb{Z}^n \times \mathbb{Z}^n$ by

$$\sigma_\hbar(p,q) = \exp(-\pi\sqrt{-1}\hbar\gamma(p,q)),$$

and then define a deformed convolution product

$$(\phi *_\hbar \psi)(p) = \sum_q \phi(q)\psi(p-q)\sigma_\hbar(q, p-q).$$

The involution on $S(\mathbb{Z}^n)$ is defined, independent of $\hbar$, as follow:

$$\phi^*(p) = \bar{\phi}(-p)$$

which, under the inverse of the Fourier transform, is just the complex conjugation on $C^\infty(T^n)$.

Define a norm $\|\cdot\|_\hbar$ on $S(\mathbb{Z}^n)$ as the operator norm for the action of $S(\mathbb{Z}^n)$ on $l^2(\mathbb{Z}^n)$ given by $\phi\cdot\xi = \phi * \hbar\xi$. We define $C_\hbar$ to be $C^\infty(T^n)$ with the product, the involution, and the norm obtained by pulling back, via the Fourier transform, the product $*\hbar$, the involution, and norm $\|\cdot\|_\hbar$ we defined above. Then $\{C_\hbar\}_{\hbar\in\mathbb{R}}$ is a strict deformation quantization of the Poisson manifold (T^n, Θ).

A_Θ is defined as C_1, the algebra for $\hbar = 1$. Then, by definition, $C_\hbar = A_{\hbar\Theta}$. An easy computation shows that

$$U_k U_j = \exp(2\pi \mathrm{i}\theta_{jk}) U_j U_k,$$

where $U_i = \exp(2\pi\sqrt{-1}x_i)$. The enveloping C^*-algebra $\bar{A}_\Theta$ of A_Θ is the universal C^*-algebra generated by n unitary operators satisfying the above relations. $\bar{A}_\Theta$ is called *the non-commutative torus*. The non-commutative tori appear naturally in M-theory compactification [3].

We define a $\mathbb{Z}_2$-action on T^n by

$$\gamma\cdot(x_1,\dots,x_n) = (-x_1,\dots,-x_n),$$

where γ is the non-identity element of $\mathbb{Z}_2$. (From now on, γ will denote the non-identity element of $\mathbb{Z}_2$.) This $\mathbb{Z}_2$-action on (T^n, Θ) preserves the Poisson bracket, i.e.,

$$\{f^\gamma, g^\gamma\} = \{f, g\}^\gamma,$$

where f^γ is defined as $f^\gamma(x) = f(-x)$. Also, the strict deformation quantization of $(T^n - \Theta)$ defined as above is invariant under the $\mathbb{Z}_2$-action, i.e.,

$$f^\gamma *_\hbar g^\gamma = (f *_\hbar g)^\gamma.$$

Hence the strict deformation quantization of (T^n, Θ) restricts to a strict deformation quantization of the Poisson orbifold $T^n/\mathbb{Z}_2$, which is flabby. A smooth function f on $T^n/\mathbb{Z}_2$

is just an smooth even function on T^n, i.e., $f(-x) = f(x)$. This strict deformation quantization is given by $\{A^{\sigma}_{\hbar\Theta}\}_{\hbar\in\mathbb{R}}$. A^{σ}_{Θ} denotes the subalgebra of A_{Θ} which consists of even functions in A_{Θ}. Its closure $\bar{A}^{\sigma}_{\Theta}$ in $\bar{A}_{\Theta}$ consists of even functions in $\bar{A}_{\Theta}$. $\bar{A}^{\sigma}_{\Theta}$ is called the symmetrized non-commutative torus.

We will simply write U_p for $\exp(2\pi\sqrt{-1}p\cdot x)$. Note that $(U_p)^{\gamma} = U_{-p} = (U_p)^*$. The difference between the action by γ and the $*$-operation is that the former is linear but the latter is conjugate-linear. Then the dense subalgebra A^{σ}_{Θ} of $\bar{A}^{\sigma}_{\Theta}$ consists of linear combinations of $\{U_p + U_{-p} | p \in \mathbb{Z}^n\}$.

Theorem 1. *Assume that there exists an entry θ_{jk} of Θ such that $4\theta_{jk}$ is not an integer. Then the symmetrized non-commutative torus $\bar{A}^{\sigma}_{\Theta}$ is Morita-equivalent to $\bar{A}_{\Theta} \rtimes \mathbb{Z}_2$.*

Proof. (For the notion of Morita-equivalence, see [7]). We let C and D denote the algebra $C(\mathbb{Z}_2, A_{\Theta})$ and the dense subalgebra A^{σ}_{Θ} of $\bar{A}^{\sigma}_{\Theta}$, respectively, where $C(\mathbb{Z}_2, A_{\Theta})$ is the set of maps from $\mathbb{Z}_2$ to A_{Θ}. Recall that the product on $C(\mathbb{Z}_2, A_{\Theta})$ is given as follows:

$$(\Lambda\Psi)(e) = \Lambda(e)\Psi(e) + \Lambda(\gamma)\Psi(\gamma))^{\gamma}, \qquad (\Lambda\Psi)(\gamma) = \Lambda(e)\Psi(\gamma) + \Lambda(\gamma)\Psi(e))^{\gamma},$$

where e it the additive identity of $\mathbb{Z}_2$. For a C–D bimodule, we take $\mathcal{E} = A_{\Theta}$. The right D-module structure on $\mathcal{E}$ is given by right multiplications. A D-valued inner product on $\mathcal{E}$ is defined by

$$\langle U, V\rangle_D = U^*V + (U^*)^{\gamma}V^{\gamma}.$$

The left C-module structure on $\mathcal{E}$ is given as follows: for $\Psi \in C$, $U \in \mathcal{E}$,

$$\Psi \cdot U = \Psi(e)U + \Psi(\gamma)U^{\gamma}.$$

We define a C-valued inner product

$$\langle U, V\rangle_C(e) = UV^*, \qquad \langle U, V\rangle_C(\gamma) = U(V^*)^{\gamma},$$

where e is the identity element of $\mathbb{Z}_2$. Easily, we have

$$\langle U, V\rangle_C \cdot W = U \cdot \langle V, W\rangle_D,$$

which is one of the requirements in the definition of Morita-equivalence.

We proceed to prove that the linear span $\langle\mathcal{E}, \mathcal{E}\rangle_C$ of $\{\langle x, y\rangle_C | x, y \in \mathcal{E}\}$ is all of C. Since $\langle\mathcal{E}, \mathcal{E}\rangle_C$ is not just a vector space but an ideal of C, we only need to show that the identity element Φ_0 of C lies in $\langle\mathcal{E}, \mathcal{E}\rangle_C$, where the identity element Φ_0 is given by $\Phi_0(e) = 1$ and $\Phi_0(\gamma) = 0$. By the assumption, we have an entry θ_{jk} such that $4\theta_{jk}$ is not an integer. We define an element $\Lambda \in C$ by $\Lambda(e) = U_j^{-2}$, $\Lambda(\gamma) = -U_j^{-2}U_k^2U_j^2$. Then we have

$$\Lambda(\langle U_j, U_j^{-1}\rangle_C - \langle U_k, U_k^{-1}\rangle_C + \langle U_k^2, 1\rangle_C) = (1 - \mathrm{e}^{8\pi\sqrt{-1}\theta_{jk}})\Phi_0.$$

Since $1 - \exp(8\pi\sqrt{-1}\theta_{jk})$ is different from 0, the identity element Φ_0 lies in $\langle\mathcal{E}, \mathcal{E}\rangle_C$. Therefore $\langle\mathcal{E}, \mathcal{E}\rangle_C$ is dense in $\bar{A}_{\Theta} \rtimes \mathbb{Z}_2$.

It is clear that $\langle \mathcal{E}, \mathcal{E} \rangle_D$ is dense in $\bar{A}^\sigma_\Theta$. Indeed, for any $p \in \mathbb{Z}^n$,

$$\langle 1, U_{-p} \rangle_D = U_p + U_{-p}.$$

The inequalities required in the definition of Morita-equivalence are also clearly satisfied. Therefore, $\mathcal{E}$ completes into a Morita-equivalence bimodule between $\bar{A}_\Theta \rtimes \mathbb{Z}_2$ and $\bar{A}^\sigma_\Theta$. □

Hence $\bar{A}_\Theta \rtimes \mathbb{Z}_2$ and $\bar{A}^\sigma_\Theta$ have the same K-theory groups, provided Θ satisfies the assumption in the above theorem. Therefore, we have the following theorem, which was proved for the case of Θ being totally irrational [4].

Theorem 2. *Assume that Θ has an entry θ_{jk} such that $4\theta_{jk}$ is not an integer. Then*

$$K_0(\bar{A}^\sigma_\Theta) = \mathbb{Z}^{3 \cdot 2^{n-1}}, \qquad K_1(\bar{A}^\sigma_\Theta) = 0.$$

Proof. Since $K_0(\bar{A}_\Theta) = K_1(\bar{A}_\Theta) = \mathbb{Z}^{2^{n-1}}$ [9] for any real skew-symmetric matrix Θ, the same reasoning as in Theorem 7 of [4] also works for this case. □

For an abelian group G, we define $\mathrm{rk}(G)$ as the rank of the free abelian group $G/\mathrm{tor}(G)$. Here $\mathrm{tor}(G)$ denotes the torsion subgroup of G.

Theorem 3. $\mathrm{rk}(K^0(T^4/\mathbb{Z}_2))$ *is greater than* 24.

Proof. The space $T^4/\mathbb{Z}_2$ has 16 singularities, which we enumerate by $p_1, \dots, p_{16}$. If we blow them up, we obtain a $K3$ surface Z. (For $K3$ surfaces and their significance in string theory, see [1].) We let X_k be the space obtained from $T^4/\mathbb{Z}_2$ by blowing up the first k points $p_1, \dots, p_k$. Hence $X_{16} = Z$. We consider the pair (S^3, Z), where S^3 is the 3-sphere to which the point p_{16} has been blown up. Then the one point compactification of $Z - S^3$ is X_{15}. We consider the following 6-term exact sequence in K-theory [5]

$$\begin{array}{ccccc} K^0(Z - S^3) & \xrightarrow{q_0^*} & K^0(Z) & \xrightarrow{i_0^*} & K^0(S^3) \\ \partial_1 \uparrow & & & & \downarrow \partial_0 \\ K^1(S^3) & \xleftarrow{i_1^*} & K^1(Z) & \xleftarrow{q_1^*} & K^1(Z - S^3). \end{array}$$

From $H^{\mathrm{ev}}(Z, \mathbb{Q}) \cong \mathbb{Q}^{24}$ and $H^{\mathrm{odd}}(Z, \mathbb{Q}) = 0$ [2], we have

$$\mathrm{rk}(K^0(Z)) = 24, \qquad \mathrm{rk}(K^1(Z)) = 0.$$

Since $\mathrm{im}\, i_1^*$ is a subgroup of $K^1(S^3) \cong \mathbb{Z}$, $\mathrm{im}\, i_1^*$ must be 0. It means that ∂_1 is injective. Therefore $\ker q_1^* \cong \mathbb{Z}$. Since $K^0(Z)/\mathrm{im}\, q_1^*$ is isomorphic to a subgroup of $K^0(S^3) \cong \mathbb{Z}$, we have $\mathrm{rk}(\mathrm{im} q_1^*) \geq \mathrm{rk}(K^0(Z)) - 1 = 23$. The fact that $\mathrm{rk}(\ker q_1^*) = 1$ implies that $\mathrm{rk}(K^0(Z - S^3) \geq 24$. Consequently, we have

$$\mathrm{rk}(K^0(X_{15})) = \mathrm{rk}(K^0(Z - S^3) \oplus \mathbb{Z}) \geq 25.$$

Now, for $k = 1, \dots, n-1$, we consider the pair (S^3, X_k), where S^3 is the 3-sphere to which the point p_k has been blown up. Then the exact sequence

$$K^0(X_k - S^3) \to K^0(X_k) \to K^0(S^3) \cong \mathbb{Z}$$

gives us the inequality $\mathrm{rk}(K^0(X_k - S^3)) \geq \mathrm{rk}(K^0(X_k)) - 1$. Since X_{k-1} is the one-point compactification of $X_k - S^3$, we have

$$\mathrm{rk}(K^0(X_{k-1})) \geq \mathrm{rk}(K^0(X_k)).$$

Hence it follows that $\mathrm{rk}(K^0(T^4/\mathbb{Z}_2)) \geq 25$. □

Remark. Let Θ be a non-zero 4×4 real skew-symmetric matrix. Then we can find a number s such that $s\Theta$ satisfies the assumption of Theorem 1. Hence $K_0(\bar{A}^{\sigma}_{s\Theta}) = \mathbb{Z}^{24}$, which is different from $K_0(\bar{A}^{\sigma}_{0\cdot\Theta}) = K_0(C(T^4/\mathbb{Z}_2))$. Therefore, the flabby strict deformation quantization $\{A^{\sigma}_{t\Theta}\}_{t\in\mathbb{R}}$ of the Poisson orbifold $T^4/\mathbb{Z}_2$ gives us an example, where $K^*(\bar{A}^{\sigma}_{t\Theta})$ varies as t varies.

Acknowledgements

This work was supported by the Brain Korea 21 Project.

References

[1] P. Aspinwall, $K3$ surfaces and string duality. hep-th/9611137.
[2] W. Barth, C. Peters, A. Van de Ven, Compact Complex Surfaces, Springer, Berlin, 1984.
[3] A. Connes, M. Douglas, A. Schwarz, Noncommutative geometry and matrix theory: compactification on tori, J. High Energy Phys. 2 (1998) 3.
[4] C. Farsi, N. Watling, Symmetrized non-commutative tori, Math. Ann. 296 (1993) 739–741.
[5] M. Karoubi, K-theory, Springer, Berlin, 1978.
[6] M. Kontsevich, Deformation quantization of Poisson manifolds I. q-alg/9709040.
[7] I. Raeburn, D. Williams, Morita Equivalence and Continuous Trace C^*-algebras, Mathematical Surveys and Monographs, Vol. 60, American Mathematical Society, Providence, 1998.
[8] M.A. Rieffel, Non-commutative tori—a case study of non-commutative differential manifolds, Contemp. Math. 105 (1990) 191–211.
[9] M.A. Rieffel, Projective modules over higher-dimensional non-commutative tori, Can. J. Math. XL (2) (1988) 257–338.
[10] M.A. Rieffel, Questions on quantization, Contemp. Math. 228 (1998) 315–326.